Slow motion of a paraboloid of revolution in a rotating fluid

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SUMMARY

The slow uniform motion, after an impulsive start from relative rest, of a paraboloid of revolution along the axis of a rotating fluid is investigated by using a perturbation method. The principal purpose of the note is to illustrate the mechanism by which the fluid is not subjected to any substantial radial displacement, which is a direct consequence of the requirement that the circulation round material circuits should be constant when the perturbation velocities due to the motion of the paraboloid remain small. It appears that the mechanism is an oscillatory one in which the distance between any fluid particle and the axis of rotation oscillates sinusoidally in time with small amplitude. As time progresses, the amplitude of the oscillation decays to zero everywhere except on the paraboloid. The ultimate motion is then a rigid body rotation everywhere except on the paraboloid and the axis of rotation, where the perturbation velocities continue to oscillate indefinitely with small amplitude.

1. INTRODUCTION

The motion of bodies in a rotating fluid has been a subject for a series of investigations in recent years. The perturbation caused by the motion of a body in an inviscid fluid exhibits different characteristics according as the fluid is at rest at infinity or is rotating about an axis there. Thus if the fluid is at rest at infinity, the flow is everywhere irrotational and dependent only on the instantaneous velocity of the body. But if the fluid is rotating about an axis, the perturbation in the fluid velocity depends not only on the instantaneous velocity of the body but also on its past history and is in general neither steady nor irrotational; and even in cases where a steady solution of the governing equation can be found, there is no guarantee that the flow can be set up by starting the body from rest relative to the rotating system. For these reasons it is necessary to consider an initial-value problem while dealing with this type of fluid motion.

When the body moves slowly it has been customary to use a small perturbation theory. Using this method Stewartson (1952, 1953) has investigated the slow uniform motion, after an impulsive start from relative rest, of a sphere and an ellipsoid along the axis of a rotating liquid. In both these cases he found that ultimately the fluid inside the circumscribing cylinder C with its generators parallel to the axis of rotation is pushed along in front of the body as if it were solid, while outside the cylinder there is a shearing motion parallel to the axis of rotation. There is also a swirling motion about the axis inside the cylinder C. The ultimate velocity distribution in the fluid is in general steady and two-dimensional (in the sense that the motion is the same in all planes perpendicular to the axis of rotation) everywhere except on the body and on its axis, where it oscillates finitely. In fact the linearized equations show that every slow and steady motion must also be two-dimensional.

There is, however, an *a priori* difficulty with any theory which supposes that the perturbation remains small. Since the circulation round any material circular circuit concentric with the axis must remain constant, the radius of such a circuit must always be nearly equal to its initial radius. At first sight this restriction on the total strain of the fluid seems unlikely to be satisfied since it is inconsistent with any prolonged general streaming (however small) past the body. Nevertheless, Stewartson's (1952, 1953) solutions do show that the perturbation can remain small, and these solutions must therefore contain an explanation of the mechanism by which the circulation is maintained at a constant level, though Stewartson does not point this out. Moreover, his solution is very complicated, largely due to the formation of the singular surface at the cylinder C, and this rather obscures the mechanism of the flow. In this note, therefore, a much simpler solution which does not have a surface corresponding to C is obtained, the primary aim being to illustrate the mechanism by which the circulation remains constant even when the perturbation remains small.

The flow considered is that due to the slow uniform motion, started impulsively from relative rest, of a paraboloid of revolution along the axis of a rotating liquid. It is found that the radius of any material circuit concentric with the axis of rotation executes small oscillations which are 180° out of phase with the corresponding oscillations in the azimuthal velocity component. As time progresses, the amplitude of these oscillations decays to zero everywhere except on the paraboloid. The ultimate flow is then steady and two-dimensional everywhere except on the paraboloid and on the axis of rotation. On the paraboloid the velocity oscillates finitely and on the axis the velocity component parallel to the axis oscillates finitely and the other components are zero. The swirling motion about the axis found in the case of the sphere and ellipsoid is absent here.

It may be pointed out that the results obtained here can be deduced from Stewartson's (1953) solution for an ellipsoid by carrying out the usual limiting process. But the procedure adopted here is found to be simpler.

2. Solution to the problem

We choose cylindrical polar coordinates, Oz along the axis of rotation and (r, θ) polar coordinates in a plane normal to Oz. Let the unperturbed motion of the fluid consist of a uniform angular velocity Ω about the z-axis.

A paraboloid of revolution (whose axis of symmetry coincides with the z-axis) impulsively starts to move along the axis at $t = 0$ with uniform velocity V. If we choose the origin of coordinates to be in the body, we have in effect superposed a uniform velocity $-V$ on the system and brought the body to rest. Let the components of the fluid velocity along the directions of increasing r, θ, z, be u, $\Omega r + v$, w, respectively, where u, v, w are small. Then the linearized equations of motion are

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\Omega v &= -\frac{\partial P}{\partial r}, \\ \frac{\partial v}{\partial t} + 2\Omega u &= 0, \\ \frac{\partial w}{\partial t} &= -\frac{\partial P}{\partial z}, \end{aligned}\right\} \tag{2.1}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{2.2}$$

where

$$P = \frac{p}{\rho} - \frac{1}{2}\Omega^2 r^2.$$

The boundary conditions are that

$$u \to 0, \quad v \to 0, \quad w \to -V, \quad \text{as } z \to \infty \text{ for fixed } r, t, \tag{2.3}$$

and, on the body, the component of the fluid velocity normal to the body is zero.

As Morgan (1953) pointed out, the initial disturbance travels with infinite velocity and the initial motion relative to the rotating system must be the irrotational motion with the given boundary conditions. Taking the velocity potential of this irrotational flow to be

$$\phi(r, z) = -Vz + \chi(r, z),$$

we have at $t = 0$

$$u = \frac{\partial \chi}{\partial r}, \quad v = 0, \quad w = -V + \frac{\partial \chi}{\partial z}.$$

Now to take the Laplace transforms of u, v, w, and P, we put

$$\bar{u} = \int_0^\infty e^{-st} u(r, z, t)\, dt, \quad \text{etc.}$$

Then (2.1) and (2.2) become

$$\left.\begin{aligned} s\bar{u} - 2\Omega \bar{v} &= -\frac{\partial \bar{N}}{\partial r}, \\ s\bar{v} + 2\Omega \bar{u} &= 0, \\ s\bar{w} + V &= -\frac{\partial \bar{N}}{\partial z}, \end{aligned}\right\} \tag{2.4}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\bar{u}) + \frac{\partial \bar{w}}{\partial z} = 0, \tag{2.5}$$

where
$$\bar{N} = \bar{P} - \chi,$$
and the boundary conditions (2.3) become
$$\bar{u} \to 0, \qquad \bar{v} \to 0, \qquad \bar{w} \to -\frac{V}{s} \qquad \text{as } z \to \infty. \tag{2.6}$$
Solving (2.4) we get
$$\left.\begin{aligned} \bar{u} &= -\frac{s}{s^2+4\Omega^2}\frac{\partial \bar{N}}{\partial r}, \\ \bar{v} &= \frac{2\Omega}{s^2+4\Omega^2}\frac{\partial \bar{N}}{\partial r}, \\ \bar{w} &= -\frac{V}{s} - \frac{1}{s}\frac{\partial \bar{N}}{\partial z}, \end{aligned}\right\} \tag{2.7}$$
so that the continuity condition (2.5) becomes
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{N}}{\partial r}\right) + \frac{s^2+4\Omega^2}{s^2}\frac{\partial^2 \bar{N}}{\partial z^2} = 0, \tag{2.8}$$
and the boundary condition (2.6) becomes
$$\frac{\partial \bar{N}}{\partial z} \to 0 \quad \text{as } z \to \infty. \tag{2.9}$$

If the section of the paraboloid in the (z, r)-plane is $z = -ar^2$, the condition on the body is
$$2aru + w = 0, \qquad \text{or } 2ar\bar{u} + \bar{w} = 0,$$
or, in view of (2.7),
$$2ar\frac{s^2}{s^2+4\Omega^2}\frac{\partial \bar{N}}{\partial r} + \frac{\partial \bar{N}}{\partial z} = -V. \tag{2.10}$$
So equation (2.8) is to be solved with boundary conditions (2.9) and (2.10).

Now we can easily formulate the problem in a coordinate system in which (2.8) can be solved simply and in which the body is a coordinate surface, by taking a suitable transformation of independent variables. We introduce new coordinates (ξ, η) defined by
$$z + \frac{K^2}{4a} = K(\xi^2 - \eta^2), \qquad r = 2\xi\eta, \tag{2.11}$$
where
$$K^2 = \frac{s^2+4\Omega^2}{s^2}.$$
On the paraboloid we have
$$\xi = \xi_0 = (K/4a)^{1/2}.$$
With the above transformation equation (2.8) becomes
$$\frac{\partial^2 \bar{N}}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial \bar{N}}{\partial \xi} + \frac{\partial^2 \bar{N}}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \bar{N}}{\partial \eta} = 0, \tag{2.12}$$
and the boundary condition (2.10) becomes
$$\frac{\partial \bar{N}}{\partial \xi^2} = -2KV\xi_0 \quad \text{on } \xi = \xi_0. \tag{2.13}$$

The appropriate solution of (2.12) is

$$\bar{N} = (A + B\log\xi)(C + D\log\eta),$$

and using (2.13) we get

$$\bar{N} = -2KV\xi_0\log\xi. \tag{2.14}$$

Now

$$\frac{\partial\bar{N}}{\partial z} = \frac{1}{2K(\xi^2+\eta^2)}\left(\xi\frac{\partial\bar{N}}{\partial\xi} - \eta\frac{\partial\bar{N}}{\partial\eta}\right) = -\frac{V\xi_0^2}{\xi^2+\eta^2}.$$

Therefore $\partial\bar{N}/\partial z \to 0$ as $\xi \to \infty$, in agreement with (2.9). Thus (2.14) is the appropriate solution.

The results for $\bar{u}$, $\bar{v}$, $\bar{w}$ follow immediately from (2.7). Finally, inverting these Laplace transforms, we find that the velocity components at any point of the fluid are given by

$$\left.\begin{aligned} u &= \frac{V}{8ar\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(1-\frac{(1+4az)s^2+4\Omega^2}{\omega^2}\right)\frac{e^{st}}{s}\,ds,\\ v &= -\frac{\Omega V}{4ar\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\left(1-\frac{(1+4az)s^2+4\Omega^2}{\omega^2}\right)\frac{e^{st}}{s^2}\,ds,\\ w &= -V+\frac{V}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{s^2+4\Omega^2}{\omega^2}\,e^{st}\,ds, \end{aligned}\right\} \tag{2.15}$$

where $\gamma > 0$ and

$$\left.\begin{aligned} \omega^2 &= \{(1+4az)^2+16a^2r^2\}^{1/2}(s^2+4\Omega^2l_1^2)^{1/2}(s^2+4\Omega^2l_2^2)^{1/2},\\ l_1^2 &= \frac{1+4az+8a^2r^2-8ar(a^2r^2+az)^{1/2}}{(1+4az)^2+16a^2r^2},\\ l_2^2 &= \frac{1+4az+8a^2r^2+8ar(a^2r^2+az)^{1/2}}{(1+4az)^2+16a^2r^2}. \end{aligned}\right\} \tag{2.16}$$

These results represent a complete formal solution to the problem. For the present purpose of ascertaining the general features of the flow, however, it is only necessary to consider certain special cases of the formulae (2.15).

3. General features of the flow

On the surface of the paraboloid the integrals (2.15) simplify considerably and it is possible to evaluate them in the forms

$$\left.\begin{aligned} u &= \frac{2Var}{1+4a^2r^2}\cos\frac{2\Omega t}{(1+4a^2r^2)^{1/2}},\\ v &= -\frac{2Var}{(1+4a^2r^2)^{1/2}}\sin\frac{2\Omega t}{(1+4a^2r^2)^{1/2}},\\ w &= -\frac{4Va^2r^2}{1+4a^2r^2}\cos\frac{2\Omega t}{(1+4a^2r^2)^{1/2}}. \end{aligned}\right\} \tag{3.1}$$

Thus we find that the motion never becomes steady on the paraboloid. More important, perhaps, these results show very simply the way in which the circulation round circular material circuits, concentric with the axis of rotation and lying on the paraboloid, remains constant. Since the radial velocity u oscillates sinusoidally in time, the radius of such a circuit must oscillate in a similar way, and the primary rotation then makes an oscillatory contribution to the circulation round the circuit. This must, in turn, be counterbalanced by an oscillatory contribution from the azimuthal perturbation velocity v, in accordance with (3.1). As far as the validity of the linearized analysis is concerned, the essential feature of this mechanism is that no fluid particle is displaced appreciably in a radial direction from its initial position.

Similarly, on the axis of rotation we find

$$u = 0, \qquad v = 0, \qquad w = -\frac{4Vaz}{1+4az}\cos\frac{2\Omega t}{(1+4az)^{1/2}}. \tag{3.2}$$

Here again the oscillatory axial velocity implies that small material circuits surrounding the axis of rotation are never swept on to the surface of the paraboloid, thereby increasing their perimeter by a large factor; an essential result if the azimuthal perturbation velocity is to remain small.

From these special cases it is reasonable to infer that the same oscillatory mechanism is responsible for maintaining the radial positions of all fluid particles, and this may be verified directly when the motion is approaching its ultimate form. Thus, for large values of t, the integrals in (2.15) may be evaluated by inserting cuts in the s-plane from $s = \pm 2i\Omega l_1$ and $s = \pm 2i\Omega l_2$ along lines on which the imaginary part of s is constant and the real part decreases. The path of integration may now be replaced by a path round the infinite semicircle $\mathscr{R}\{s\} < 0$ and round the four cuts. For example, the contribution from the branch point $s = 2i\Omega l_1$ to the integral in the first of the equations (2.15) is found to be

$$e^{2i\Omega l_1 t}\frac{1-(1+4az)l_1^2}{l_1(l^2-l_1^2)^{1/2}(4i\Omega l_1)^{1/2}}\frac{\Gamma(\frac{1}{2})}{t^{1/2}}$$

for large t. In this way we find that

$$u \sim -\frac{V}{16(ar)^{3/2}(a^2r^2+az)^{1/4}}\left[\frac{1-(1+4az)l_1^2}{l_1(\Omega l_1)^{1/2}}\sin(2\Omega l_1 t-\tfrac{1}{4}\pi)+ \right.$$
$$\left. + \frac{1-(1+4az)l_2^2}{l_2(\Omega l_2)^{1/2}}\sin(2\Omega l_2 t+\tfrac{1}{4}\pi)\right]\frac{1}{(\pi t)^{1/2}},$$

$$v \sim \frac{V}{16(ar)^{3/2}(a^2r^2+az)^{1/4}}\left[\frac{1-(1+4az)l_1^2}{l_1^2(\Omega l_1)^{1/2}}\cos(2\Omega l_1 t-\tfrac{1}{4}\pi)+ \right.$$
$$\left. + \frac{1-(1+4az)l_2^2}{l_2^2(\Omega l_2)^{1/2}}\cos(2\Omega l_2 t+\tfrac{1}{4}\pi)\right]\frac{1}{(\pi t)^{1/2}},$$

$$w \sim \frac{V}{4(ar)^{1/2}(a^2r^2+az)^{1/4}}\left[\frac{1-l_1^2}{l_1(\Omega l_1)^{1/2}}\sin(2\Omega l_1 t-\tfrac{1}{4}\pi)+ \right.$$
$$\left. + \frac{1-l_2^2}{l_2(\Omega l_2)^{1/2}}\sin(2\Omega l_2 t+\tfrac{1}{4}\pi)\right]\frac{1}{(\pi t)^{1/2}}.$$

Thus the only significant difference here is that the amplitude of the oscillations decreases to zero, so that the ultimate motion is in general steady and two-dimensional and the axial velocity of the fluid is ultimately the same as that of the paraboloid.

In view of the ultimate singularity in the velocity gradients on the axis and body it seems that the detailed form (but probably not the general nature) of the solution is of doubtful validity in this neighbourhood.

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